Renormalization group for the XYZ model

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Abstract

We study in a rigorous way the XYZ spin model by Renormalization Group methods.

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The XYZ model is exactly solvable [1] by the *transfer matrix formalism* as it is equivalent [2] to the eight vertex model. The solution is so complicated that it is very difficult to compute the correlation functions from it (an attempt with some preliminary results is in [3]). The correlations are then computed in an approximate way by linearizing the bands and taking the continuum limit [4], so introducing spurious u.v. divergences. One has to introduce an *ad hoc* cut-off, absent in the original model, for applying the *bosonization methods*, and it is not very clear the relationship of the obtained correlations with the real ones. Finally *Bethe ansatz* and the *conformal algebra* methods cannot be applied to the XYZ chain but only to its limiting case given by the XXZ chain [5] (also bosonization results are mainly for this case).

In this letter we apply to the XYZ chain the RG methods developed for QFT [6]; we show that, for small anisotropy and J_3, the correlations can be expressed by convergent series. This is almost equivalent to know the correlations exactly, as one can compute the first orders having a rigorous bound on the remainder *i.e.* the correlations are known up to a small error. The comparison of the so obtained correlations with experiments or numerics is then without ambiguity. No approximation are necessary in our approach, which can be extended to a variety of models. Bounds on the corrections due to the finite size effect are naturally obtained, as our results are *uniform* in the size. We write the XYZ chain as a system of interacting fermions, and we compute the fermionic two point different (imaginary) time correlation function. The computation of the other correlations is a straightforward (but cumbersome) consequence of our results.

If $(S^1_x, S^2_x, S^3_x) = \frac{1}{2}(\sigma^1_x, \sigma^2_x, \sigma^3_x)$, σ^α_x, $\alpha = 1, 2, 3$ being the Pauli matrices and $x = 1, 2, ..., N$ the hamiltonian H is

$$\sum_{x=1}^N [J_1 S^1_x S^1_{x+1} + J_2 S^2_x S^2_{x+1} + J_3 S^3_x S^3_{x+1} + h S^3_x] + U^1_N$$

where the last term is a boundary term and $S^3_{N+1} = S^3_1$ (the boundary conditions on S^1_x, S^2_x will be specified later fixing U^1_N)). The *anysotropy* is $u = \frac{J_1 - J_2}{J_1 + J_2} > 0$ for fixing ideas.

By a *Jordan-Wigner transformation* [7] we write, if $S^\pm_x = S^1_x \pm i S^2_x$, $S^-_x = e^{-i\pi \sum_{y=1}^{x-1} \psi^+_y \psi^-_y} \psi^-_x$ and $S^+_x = \psi^+_x e^{i\pi \sum_{y=1}^{x-1} \psi^+_y \psi^-_y}$, where $\psi^\pm_x$ are fermionic operators. Moreover $S^3_x = \psi^+_x \psi^-_x - \frac{1}{2}$. Fixing $J_1 + J_2 = 1$, H is:

$$\sum_{x=1}^N \{[\psi^+_x \psi^-_{x+1} + \psi^+_{x+1} \psi^-_x] + u[\psi^+_x \psi^+_{x+1} + \psi^-_{x+1} \psi^-_x] +$$

$$J_3(\psi^+_x \psi^-_x - \frac{1}{2})(\psi^+_{x+1} \psi^-_{x+1} - \frac{1}{2}) + h(\psi^+_x \psi^-_x - \frac{1}{2})\} + U^2_N$$

where U_N^2 is a boundary term. We choose U_N^1 so that $U_N^2 = 0$ and the fermions verify periodic boundary conditions (in [7] this choice for the XY chain is called "c-cyclic"). This is achieved by setting $U_N^1 = [(-S_N^+ S_{N+1}^- + S_N^+ e^{i\pi\mathcal{N}} S_1^-) + c.c.] + u[(-S_N^+ S_{N+1}^+ + S_N^+ e^{i\pi\mathcal{N}} S_1^-) + c.c.]$, if $\mathcal{N} = \sum_{x=1}^N \psi_x^+ \psi_x$; as $[(-1)^{\mathcal{N}}, H] = 0$ the eigenvectors of H are divided in two subspaces on which $(-1)^{\mathcal{N}}$ is equal to 1 or -1 and on such subspaces U_N^1 is a boundary term. Let be $S^{\varepsilon_1,\varepsilon_2} S(\vec{x}) = \lim_{N,\beta\to\infty} < T\psi_{\vec{x}}^{\varepsilon_1}\psi_{0}^{\varepsilon_2} >_{N,\beta}$, $\varepsilon_i = \pm$, T is the time ordering and $<\cdot>= \frac{\mathrm{tr} e^{-\beta H}\cdot}{\mathrm{tr} e^{-\beta H}}$. We prove that, if $\lim_{N,\beta\to\infty} S_{N,\beta}^{\varepsilon_1,\varepsilon_2}(\vec{x}) = S^{\varepsilon_1,\varepsilon_2}(\vec{x})$, $\vec{x} = x, t$ and setting $p_F = \cos^{-1}(h - J_3) \neq n\pi$, $v_0 = \sin p_F$ and $|\vec{x}| = \sqrt{x^2 + v_0^2 x_0^2}$, $\vec{k} = k, k_0$:

For J_3, u suitably small

$$S^{\varepsilon_1,\varepsilon_2}(\vec{x}) = S_0^{\varepsilon_1,\varepsilon_2}(\vec{x}) + \varepsilon S_1^{\varepsilon_1,\varepsilon_2}(\vec{x})$$

where $S_0^{-,+}(\vec{x})$ and $S_0^{+,+}(\vec{x})$ are respectively

$$\int d\vec{k} \sin(\bar{p}_F x + k_0 t) \frac{2ie^{ikx}}{Z(k)} \frac{ik_0 - v_0 \sin(|k| - \bar{p}_F)}{k_0^2 + v_0^2 \sin^2(|k| - \bar{p}_F) + \sigma(k)^2}$$

$$\int d\vec{k} \sin(\bar{p}_F x) \frac{2ie^{i\vec{k}\vec{x}}}{Z(k)} \frac{\sigma(k)}{k_0^2 + v_0^2 \sin^2(|k| - \bar{p}_F) + \sigma(k)^2}$$

where $\varepsilon = \max(u, |J_3|, u^{1+\eta_2})$, $\bar{p}_F = p_F + O(J_3)$ and $Z(k), \sigma(k)$ are smooth functions such that $|Z(k) - 1| \leq \varepsilon$, $|\sigma(k) - u| \leq \varepsilon$ for $||k| - \bar{p}_F| \geq \min[(\bar{p}_F/2, (\pi - \bar{p}_F)/2]$ and

$$Z(\pm\bar{p}_F) \equiv \hat{Z} = u^{-\eta_1} \qquad \sigma(\pm\bar{p}_F) \equiv \hat{\sigma} = u^{1+\eta_2}$$

with $\eta_1 = \beta_1 (J_3)^2 + O((J_3)^3)$, $\eta_2 = -\beta_2 J_3 + O((J_3)^2)$, $\beta_1, \beta_2 > 0$. Moreover for $|\vec{x}| \geq \hat{u}^{-1}$, $S_{0,1}^{\varepsilon_1,\varepsilon_2}(\vec{x})$ have a long distance faster than any power decay, i.e. for any M, $|S_{0,1}^{\varepsilon_1,\varepsilon_2}(\vec{x})| \leq \frac{C_M}{\hat{Z}} \frac{\hat{u}}{1 + (\hat{u}|\vec{x}|)^M}$ if C_M is a constant; for $1 \leq |\vec{x}| \leq \hat{u}^{-1}$ they have a transient slow decay $|S_{0,1}^{\varepsilon_1,\varepsilon_2}(\vec{x})| \leq \frac{C_1}{|\vec{x}|^{1+\eta_3}}$, with $\eta_3 = \eta_1(1 + \eta_2)^{-1}$.

The optimal bound for $S_{0,1}^{\varepsilon_1,\varepsilon_2}(\vec{x})$ should be $|S_{0,1}^{\varepsilon_1,\varepsilon_2}(\vec{x})| \leq \frac{c_1}{|\vec{x}|^{1+\eta_3}} e^{-c_2 u^{1+\eta_2}|\vec{x}|}$, $c_1, c_2 > 0$ constants, and this could be proved by a slight improvement of our techniques. We see that there is an *anomalous gap* and an *anomalous wave function renormalization.* Moreover the oscillation period $\bar{p}_F$ depends on J_3. The explicit form of $Z(k)$ and $\sigma(k)$ will be given below; if $u = 0$ $\sigma(k) = 0$ and $\hat{Z}(k) = |k|^{-\eta_3}$ according with the expected power law decay of the XXZ chain. A simple consequence of our analysis is that, in the $J_3 = h = 0$ case, $| < S_{\vec{x}}^3 S_0^3 > - < S_{\vec{x}}^3 >< S_0^3 > | \leq \frac{c_1}{|\vec{x}|^2} e^{-c_2 u|\vec{x}|}$ in agreement with the results in the XY chain in [7], for small u and $t = 0$ $\sin^2(\frac{\pi}{2}x) \frac{-1}{4\pi} (\frac{1-u}{1+u})^x \frac{1}{x^2}[1 + O(\frac{1}{x})]$.

RG analysis. The Grand-Canonical Schwinger function can be defined by Grassman integrals so we can write $< T \prod_{i=1}^n \psi_{\vec{x}_i}^{\varepsilon_i} >_{L,\beta}$ as

$$\frac{\int \{\mathcal{D}\psi e^{-\int d\vec{k} \psi_{\vec{k}}^+ (-ik_0 - E(k)) \psi_{\vec{k}}^-}\} e^{-V(\psi)} \prod_{i=1}^n \psi_{\vec{x}_i}^{\sigma_i}}{\int \{\mathcal{D}\psi e^{-\int d\vec{k} \psi_{\vec{k}}^+ (-ik_0 - E(k)) \psi_{\vec{k}}^-}\} e^{-V(\psi)}} \quad (1)$$

where $\vec{k} = 2\pi(\frac{n_1}{N}, \frac{n_0 + 2^{-1}}{\beta})$, if n_0, n_1 are integers, $E(k) = \cos k - \cos p_F$, $\psi_{\vec{x}}^{\pm}, \psi_{\vec{k}}^{\pm}$, are *Grassman variables* and we denote by $\int \{\mathcal{D}\psi e^{-\int dk \psi_{\vec{k}}^+ h(\vec{k})^{-1} \psi_{\vec{k}}^-}\}$ the *fermionic integration* with propagator $\int d\vec{k} e^{i\vec{k}(\vec{x}-\vec{y})} h(\vec{k})$, if $\int d\vec{k} = \frac{1}{L\beta}\sum_{\vec{k}}$. Finally $V(\psi)$ can be written as $V(\psi) = J_3 \bar{V} + uP + \nu N$ where $\bar{V} = \int \prod_{i=1}^4 d\vec{k}_i \cos(k_1 - k_2) \psi_{\vec{k}_1}^+ \psi_{\vec{k}_2}^- \psi_{\vec{k}_3}^+ \psi_{\vec{k}_4}^- \delta(\vec{k}_1 - \vec{k}_2 + \vec{k}_3 - \vec{k}_4)$ and $P = \int d\vec{k} [e^{ik} \psi_{\vec{k}}^+ \psi_{-\vec{k}}^+ + e^{-ik} \psi_{-\vec{k}}^- \psi_{\vec{k}}^-]$, $N = \int d\vec{k} \psi_{\vec{k}}^+ \psi_{\vec{k}}^-$. Note that ν is a *counterterm* to be fixed so that the Fermi momentum in the $J_3 = 0$ or $J_3 \neq 0$ theory are the same. In fact, as the oscillation period is changed by the presence of the J_3 term, we find technically convenient to fix it to its value in the $J_3 = 0$ case by adding the counterterm. According to the (formal) Luttinger theorem, this means that we add a magnetic field so that the mean magnetization in the direction of the magnetic field is the same as in the $J_3 = 0$ case.

We start by integrating the denominator of eq.(1), the *partition function* $\mathcal{N}$. We perform a decomposition of the propagator $g(\vec{k}) = f_1(\vec{k}))g(\vec{k}) + (1 - f_1(\vec{k}))g(\vec{k})$ where $f_1(\vec{k}) = 1 - \chi(k - p_F, k_0) - \chi(k + p_F, k_0)$ and χ is a smooth compact support function such that $\chi(k \pm p_F, k_0)$ are non vanishing only in two non overlapping regions around $p_F, 0$ and $-p_F, 0$ respectively. We call the two addends respectively $g^{>0}(\vec{k})$ and $g^{\leq 0}(\vec{k})$ and this allows us to represent $\psi_{\vec{k}}^{\pm}$ as sum of two independent Grassmanian variables, $\psi_{\vec{k}}^{(>0)\pm}$, $\psi_{\vec{k}}^{(\leq 0)\pm}$. The integration on $\psi^{(>0)}$ allows us to write $\mathcal{N}$ with ψ replaced by $\psi^{(\leq 0)}$ and V replaced

by V^0 defined as [8]:

$$\sum_{n=0}^{\infty}\int\prod_{i=1}^{n}d\vec{k}_i W_n^h(\vec{k}_1,\ldots,\vec{k}_n)\prod_{i=1}^{n}\psi_{\vec{k}_i}^{(\leq h)\varepsilon_i}\delta(\sum_{i=1}^{n}\varepsilon_i\vec{k}_i) \tag{2}$$

with $h=0$, where $|W_n^0|\leq C^m z^{\max(2,n-1)}$ if $z=\max(|\lambda|,|u|,|\nu|)$. Note that, unless $u=0$, $\sum_{i=1}^n\varepsilon_i\neq 0$ as $[H,\mathcal{N}]\neq 0$. We set $k=k'+\omega p_F$, $\omega=\pm 1$, $\mathrm{sign}(\omega k)>0$; moreover we write $1-f_1(\vec{k})=\sum_{\omega=\pm 1}\sum_{h=-\infty}^{0}f_h(\vec{k}')$, $f_h(\vec{k}')=\chi(\gamma^{-h}\vec{k}')-\chi(\gamma^{-h+1}\vec{k}')$ and $C_h^{-1}=\sum_{k=-\infty}^{h}f_h$. This allows us to write $\psi_k^{(\leq 0)}=\sum_{\omega=\pm 1}\sum_{h=-\infty}^{0}\psi_{k,\omega}^{(h)}$. In other words we represent the fermions by two Fermi fields with label h, with momenta close $O(\gamma^h)$ respectively to p_F or $-p_F$. We proceed iteratively setting $Z_0=1$: once the fields $\psi^{(0)},\ldots,\psi^{(h+1)}$ have been integrated $\mathcal{N}$ is given by:

$$\int\{\mathcal{D}\psi^{(\leq h)}e^{-\int d\vec{k}' C_h Z_h\bar{\psi}_{\vec{k}'}^{(\leq h)+}\mathcal{G}^{(h)}(\vec{k}')^{-1}\bar{\psi}_{\vec{k}'}^{(\leq h)-}}\} \\ e^{-V^h(\sqrt{Z_h}\psi^{(\leq h)})} \tag{3}$$

if $\bar{\psi}_{\vec{k}'}^{(\leq h)\pm}=(\psi_{\vec{k}'+p_F,1}^{(\leq h)\pm},\psi_{-\vec{k}'-p_F,-1}^{(\leq h)\mp})$ and $V^h(\psi^{(\leq h)})$ is given by terms like eq.(2). If $\alpha(k)=\cos p_F(1-\cos k)$, $[\mathcal{G}^{(h)}(\vec{k}')]_{\omega,\omega}^{-1}=(-ik_0+\omega v_0\sin\vec{k}+\omega\alpha(k'))$, $[\mathcal{G}^{(h)}(\vec{k}')]_{\omega,-\omega}^{-1}=i\omega\sigma_h(k')$. We define a *localization operator* $\mathcal{L}$ extracting the *relevant part* of the *effective potential* V^h: i) If $n>4$ then $\mathcal{L}W_n^h=0$; ii) if $n=4$ if $\delta_4^a=\delta_{\sum_{i=1}^4\varepsilon_i\omega_i p_F,0}\delta_{\sum_{i=1}^4\varepsilon_i,0}$, then $\mathcal{L}W_4^h(\vec{k}_1+\omega_1 p_F,....)=\delta_4^a W_4^h(\omega_1 p_F,...)$ iii) if $n=2$ then if $\delta_2^a=\delta_{(\omega_1-\omega_2)p_F,0}\delta_{\varepsilon_1-\varepsilon_2,0}$ and $\delta_2^b=\delta_{(\omega_1+\omega_2)p_F,0}\delta_{\varepsilon_1+\varepsilon_2,0}$ then

$$\mathcal{L}W_2^h(\vec{k}_1'+\omega_1 p_F,\vec{k}_2'+\omega_2 p_F)=\delta_2^a[W_2^h(\omega_1 p_F,\omega_2 p_F)+ \\ \omega_1 E(k'+\omega_1 p_F)\partial_k W_2^h(\omega_1 p_F,\omega_2 p_F)+k^0 \\ \partial_{k_0}W_2^h(\omega_1 p_F,\omega_2 p_F)]+\delta_2^b[W_2^h(\omega_1 p_F,\omega_2 p_F)]$$

where $E(k'+\omega p_F)=v_0\omega\sin k'+\alpha(k')$ (the symbols $\partial_k,\partial_{k_0}$ means discrete derivatives and the first deltas in $\delta_2^a,\delta_2^b,\delta_4^b$ are mod. 2π). A naive power counting argument explains why the relevant terms are only the quartic or bilinear in the fields; moreover among such terms there are still irrelevant ones *i.e.* the power counting can be improved. This is taken into account in the definition of $\mathcal{L}$, as the first of the two deltas in $\delta_4^a,\delta_2^a,\delta_2^b$ says that the relevant terms involve only fermions at the Fermi surface, *i.e.* $\sum_i\varepsilon_i\omega_i p_F=0$ *modulo* 2π, and the second takes into account that the marginal terms with $\sum_i\varepsilon_i\neq 0$ are indeed irrelevant. This will discussed below. We can write then the relevant part of the effective potential as:

$$\mathcal{L}V^h=\gamma^h n_h F_\nu^h+s_h F_\sigma^h+z_h F_\zeta^h+a_h F_\alpha^h+l_h F_\lambda^h$$

where $F_i^h=\sum_\omega\int d\vec{k}' f_i\psi_{\omega\vec{k}'+\omega p_F,\omega}^{(\leq h)+\omega}\psi_{\omega\vec{k}'+\omega p_F,\omega}^{(\leq h)-\omega}$, and $F_\sigma^h=\sum_\omega\int d\vec{k}'\psi_{\omega\vec{k}'+\omega p_F,\omega}^{(\leq h)+\omega}\psi_{-\omega\vec{k}'-\omega p_F,-\omega}^{(\leq h)-\omega}$ and F_λ^h is given by $\int[\prod_{i=1}^4 d\vec{k}_i'\psi_{\vec{k}_i'+p_F,\omega_i}^{(\leq h)\varepsilon_i}]\delta(\sum_{i=1}^4\omega_i\sigma_i\vec{k}_i')$ where $i=\nu,\alpha,\zeta$, $f_\nu=\omega$, $f_\alpha=\omega E(k'+p_F)$, $f_\zeta=-ik_0$. Moreover $l_0=J_3+O((J_3)^2)$, $s_0=u+O(uJ_3)$, $a_0,z_0=O(J_3)$, $n_0=\nu+O(J_3)$ and we have defined $\psi^{(\leq h)\pm\omega}=\psi^{(\leq 0)\pm}$ if $\omega=1$ and $\psi^{(\leq 0)\omega\pm}=\psi^{(\leq 0)\mp}$ if $\omega=-1$. We write eq.(3) as:

$$\int\mathcal{D}\psi^{(\leq h)}e^{-\int dk'\bar{\psi}_{k'}^{(\leq h)+}C_h Z_{h-1}(k')\mathcal{G}^{(h-1)}(k')^{-1}\bar{\psi}_{k'}^{(\leq h)-}} \\ e^{-\tilde{V}^h(\sqrt{Z_h}\psi^{(\leq h)})}] \tag{4}$$

where $\tilde{V}^h=\mathcal{L}\tilde{V}^h+(1-\mathcal{L})V^h$, $\mathcal{L}\tilde{V}^h=\gamma^h n_h F_\nu^h+(a_h-z_h)F_\alpha^h+l_h F_\lambda^h$ and $Z_{h-1}(k')=Z_h+C_h^{-1}Z_h z_h$, $Z_{h-1}(k')\sigma_{h-1}(k')=Z_h\sigma_h(k')+Z_h C_h^{-1}s_h$. This means that we extract from the effective potential the terms leading to a mass and wave function renormalization. Now one can perform the integration respect to $\psi^{(h)}$ rescaling the effective potential $\hat{V}^h(\psi)=\tilde{V}^h(\sqrt{\frac{Z_h}{Z_{h-1}}}\psi)$ and $\mathcal{L}\hat{V}^h=\gamma^h\nu_h F_\nu^h+\delta_h F_\alpha^h+\lambda_h F_\lambda^h$ with $\gamma^h\nu_h=\frac{Z_h}{Z_{h-1}}n_h$, $\delta_h=\frac{Z_h}{Z_{h-1}}(a_h-z_h)$, $\lambda_h=(\frac{Z_h}{Z_{h-1}})^2 l_h$ and $\vec{v}_k=\nu_h,\delta_h,\lambda_h$. The integration of $\psi^{(h)}$ has propagator $g_{\omega,\omega'}^h(x-y)=\frac{1}{Z_{h-1}}\int d\vec{k}' e^{i\vec{k}'(x-y)}\tilde{f}_h(\vec{k}')\mathcal{G}^{(h-1)}(\vec{k}')_{\omega,\omega'}$, if $Z_{h-1}\equiv Z_{h-1}(0)$ and $\tilde{f}_h=Z_{h-1}[\frac{C_h^{-1}}{Z_{h-1}(k')}-\frac{C_{h-1}^{-1}}{Z_{h-1}}]$. After this integration $\mathcal{N}$ is, up to a constant, of the form of eq.(3) with h replaced by $h-1$, and we can iterate. *In other words we have to perform a Bogolubov transformation for each scale, as the "mass" σ_h has a non trivial RG flow and it is different for any h; at the same time one has to take into account the wave function renormalization Z_h.* Let be $h^*=\inf_h\{\gamma^h\geq|\sigma_h|\}$. Note that, if h^* is finite uniformly in N,β so that $|\sigma_{h^*-1}|\gamma^{-h^*+1}\geq 1$

then $|g^{<h^*}(\vec{x})| \le \frac{1}{Z_{h^*}} \frac{C_M \gamma^{h^*}}{1+(\gamma^{h^*}|\vec{x}|)^M}$; moreover if $h \ge h^*$ we have $|g^h_{\omega,\omega}(\vec{x})| \le \frac{1}{Z_h} \frac{C_M \gamma^h}{1+(\gamma^h|\vec{x}|)^M}$ and $|g^h_{\omega,-\omega}(\vec{x})| \le \frac{1}{Z_h} \frac{|\sigma_h|}{\gamma^h} \frac{C_M \gamma^h}{1+(\gamma^h|\vec{x}|)^M}$. *The propagator for the integration of all the scales $h < h^*$ obeys to the same bound of a single scale propagator for $h \ge h^*$. Moreover for $h \ge h^*$ the bound for the non diagonal propagator has a factor more $\frac{|\sigma_h|}{\gamma^h}$ with respect to the diagonal propagator.* Finally $g^h_{\omega,\omega}(x-y) = g^h_{\omega,L}(x-y) + C^h_{1,\omega}(x-y) + C^h_{2,\omega}(x-y)$, with $g^h_{\omega,L}(x-y)$ given by $\int d\vec{k}' \frac{e^{i\vec{k}' x}}{Z_h} \frac{f(\gamma^{-2h}(k_0^2+k'^2))}{-ik_0 - 2\pi\omega k'}$ which is just the propagator "at scale h" of the Luttinger model, and the other two terms verify the bound of $g^h_{\omega,\omega}(\vec{x};\vec{y})$ with an extra factor γ^h or $\frac{|\sigma_h|}{\gamma^h}$.

Lemma*Assume that h^* is is finite uniformly in N, β and that for any $h > k \ge h^*$ there exists an ε such that $|\vec{v}_h| \le \varepsilon$ and $|\frac{\sigma_{h+1}}{\sigma_h}| \le \gamma^{c_a \varepsilon}$, $|\frac{Z_{h+1}}{Z_h}| \le \gamma^{c_b \varepsilon^2}$ with c_a, c_b postive constants. Then V^k is given by a convergent series.*

We have to show by the study of the beta function that the conditions in the lemma are verified. It is possible to choose [8],[9] the counterterm ν so that $|\nu_h| < \varepsilon$ for all $0 \ge h \ge h^*$. The Beta function can be written, for $0 \ge h \ge h^*$. :

$$\begin{aligned}
\lambda_{h-1} &= \lambda_h + G^{1,h}_\lambda + G^{2,h}_\lambda + \gamma^h R^h_\lambda \\
\sigma_{h-1} &= \sigma_h + G^{1,h}_\sigma \\
\delta_{h-1} &= \delta_h + G^{1,h}_\delta + G^{2,h}_\delta + \gamma^h R^h_\delta \\
\frac{Z_{h-1}}{Z_h} &= 1 + G^{1,h}_z + G^{2,h}_z + \gamma^h R^h_z
\end{aligned} \tag{5}$$

where a) $G^{1,h}_\lambda$, $G^{1,h}_\delta$ and $G^{1,h}_z$ depend only on $\lambda_0, \delta_0; ...\lambda_h, \delta_h$ and are given by series of terms involving only the Luttinger model part of the propagator $g^k_{\omega,L}(x-y)$, so they coincide with the Luttinger model Beta function [8],[9]; b) $G^{1,h}_\sigma$, $G^{2,h}_\lambda, G^{2,h}_\delta, G^{2,h}_z$ are given by a series of terms involving at least a propagator $C^k_{2,\omega}(x-y)$ or $g^k_{\omega,-\omega}(x-y)$ with $k \ge h$; c) R^h_i, $i = \lambda, z, \delta$ are given by a series of terms involving at least a propagator $C^k_{1,\omega}(x-y)$, $k \ge h$. By a simple computation $G^{1,h}_z = \lambda_h^2[\beta_1 + \bar{G}^h_z]$, $G^{1,h}_\sigma = \lambda_h \sigma_h[-\beta_2 + \bar{G}^h_\sigma]$, with $\beta_1, \beta_2 > 0$ and $\bar{G}^h_z, \bar{G}^h_\sigma = O(\lambda_h)$. *Moreover $G^{1,h}_\lambda$, $G^{1,h}_\delta$ coincide by definition with the Luttinger model Beta function, and it was proved in [8],[9] that it is vanishing at any order, i.e.* $G^{1,h}_\lambda(\lambda, \delta; \dots; \lambda, \delta) = 0$ and $G^{1,h}_\delta(\lambda, \delta; \dots; \lambda, \delta) = 0$. Finally as $|G^{2,h}_\lambda|, |G^{2,h}_\delta|, |G^{2,h}_z| \le K\varepsilon^2 |\sigma_h| \gamma^{-h}$, one finds, for $h \ge h^*$, $|\lambda_{h-1} - \lambda_0| < c_1 l_0^2$, $|\delta_{h-1} - \delta_0| \le c_1 l_0^2$,

$$l_0 \beta_1 c_2 h \le \log\left(\frac{\sigma_{h-1}}{\sigma_0}\right) \le l_0 \beta_1 c_3 h$$

$$-\beta_3 c_4 l_0^2 h \le \log(Z_{h-1}) \le -\beta_3 c_5 l_0^2 h$$

for suitable positive constants c_i, *i.e.* as usual in models to which the RG is succesfully applied the flow is essentially *described by the second order truncation of the beta function.* This shows that it is possible to choose J_3 so small that the conditions of the above lemma are fulfilled. We call $\eta_1 = -\log Z_{h^*} / \log u$, $1 + \eta_2 = \log \sigma_{h^*} / \log u$.

Finally as we said the integrations of the $\psi^{(<h^*)}$ is essentially *equivalent to the integration of a single scale $h \ge h^*$*, so it is well defined by the preceding arguments.

It is a standard matter to deduce an expansion for the correlations from the effective potential, and so deducing the results for $S^{\sigma_1,\sigma_2}(\vec{x})$. Moreover we call $\sigma(k), Z(k)$ respectively σ_h, Z_h for $\gamma^h \le |k| \le \gamma^{h+1}$ for $h \ge h^*$ and σ_{h^*}, Z_{h^*} for $|k| \le \gamma^{h^*}$.

Finally we discuss the modifications in the proof of the above lemma with respect to the one existing in literature for similar models,[8],[9], mainly due to the fact that $[\mathcal{N}, H] \ne 0$. V^k can be written as a sum over "Feynman graphs" obtained in following way. Let us consider n points and enclose them into a set of *clusters* v to which a scale h_v is associated; an inclusion relation is established between the clusters, in such a way that the innermost clusters are the clusters with highest scale, *i.e.* if v' is the cluster containing v then $h_{v'} < h_v$; v_0 is the largest cluster. A set of clusters can be represented as a *tree* and the set of the possible trees is denoted by $\tau_{n,k}$. To each point contained in a cluster v but not in any smaller one we associate one of the elements of $\mathcal{L}V^{h_v}$, expressed graphically as a vertex with 2 or 4 "half lines". The half lines are paired in all the possible compatible way and to the paired lines we associate a propagator $g^{h_v}_{\omega,\omega'}$ if the paired line is enclosed in the cluster v but not in any smaller one. The indices of the external lines of v are denoted by P_v and their

number by $|P_v|$. The $\mathcal{R} = 1 - \mathcal{L}$ operation acts on the cluster and its action, as we know, depend on P_v. We call V_n^k the contribution to V^k from $\tau \in \tau_{n,k}$, and $|V_n^k|(L\beta\gamma^{-kD(P_{v_0})})^{-1} \equiv |\bar{V}_n^k|$ is bounded by, if C_1 is a constant

$$C_1^n \varepsilon^n \sum_{\tau\in\tau_{n,k}} [\sum_{\{P_v\}} \prod_v \gamma^{-[D(P_v)+z(P_v)](h_v - h_{v'})}]$$

$$[\prod_{v\in V_2} |\sigma_{h_v}|\gamma^{-h_v}]$$

where $D(P_v) = -2 + |P_v|/2$, and V_2 is the set of the clusters v containing a non diagonal propagator $g^{h_v}_{\omega,-\omega}$ *i.e.* the smallest clusters containing a non diagonal propagator. The factor $z(P_v)$ (which would be zero if we had naively taken $\mathcal{R} = 1$) is defined as:1) $z(P_v) = 1$ if $|P_v| = 4$ and $\delta_4^a = 1$; 2) $z(P_v) = 2$ if $|P_v| = 2$ and $\delta_2^a = 1$; 3) $z(P_v) = 1$ if $|P_v| = 2$ and $\delta_2^b = 1$; 4)$z(P_v) = 0$ in the remaining cases. The first parenthesis is the power counting of a graph, and the second is due to the extra factor in the bound for the non diagonal propagators; the only non trivial part of the above bound is the use of the Gramm-Hadamard inequality to take into account the relative signs of the graphs (estimating them by their absolute value one would get an extra $k!$ in the bounds spoling convergence). If $D(P_v) + z(P_v) > 0$ the sum over $\tau, \{P_v\}$ can be bounded and $|\bar{V}_n^k| \le C_2^n \varepsilon^n$; however, if $\delta_4^a, \delta_2^a, \delta_2^b = 0$ this is not the case. The factors $\delta_4^a, \delta_2^a, \delta_2^b$ are products of two deltas. If the first delta is vanishing, by the support in momentum space of $g^h_{\omega,\omega'}$, it follows that there exists a fixed scale $\bar{h}$, indipendent on L, β, k, n, such that there are clusters v with the first non vanishing deltas only if $h_v \ge \bar{h}$; so these clusters give no problems (one can even choose the functions χ so that $\bar{h} \equiv 0$). Let us consider now the case in which the second deltas are non vanishing. Note that $|\sigma_{h_v}\gamma^{-h_v}| \le |\frac{\sigma_{h_v}}{\sigma_{h^*}}\frac{\sigma_{h^*}}{\gamma^{h_v}}|$ $\le \gamma^{(k-h_v)(1-O(\varepsilon))} \le \gamma^{(k-h_v)(1/2)}$ as $|\sigma_{h^*}| \le \gamma^{h_*}$.

Let us consider vertices $v_1, v_2, .., v_I$ ordered so that $v_1 > v_2 > ... > v_I$; then there exists at least a non diagonal propagator with scale h_{v_1} and

$$|\sigma_{h_{v_1}}\gamma^{-h_{v_1}}| \le \gamma^{(k-h_{v_1})(1/2)} \le \gamma^{(h_{v'_I} - h_{v_I})(1/2)}$$

$$\gamma^{(h_{v_I} - h_{v_1})(1/2)} \le \gamma^{(h_{v'_I} - h_{v_I})(1/2)}\gamma^{(h_{v'_{I-1}} - h_{v_1})(1/2)}...$$

so that $\prod_{v\in V_2} |\sigma_{h_v}|\gamma^{-h_v} \le \prod_{v\in V_b} \gamma^{-(1/2)(h_v - h_{v'})}$, if V_b are the clusters not verifing the second deltas.

At the end

$$[\prod_v \gamma^{-[D(P_v)+z(P_v)](h_v-h_{v'})}][\prod_{v\in V_2} |\sigma_{h_v}|\gamma^{-h_v}] \le$$

$$[\prod_v \gamma^{-(1/2)[D(P_v)+\bar{z}(P_v)](h_v-h_{v'})}]$$

with $D(P_v) + \bar{z}(P_v) > 0$, for any v. Hence a $C^n\varepsilon^n$ bound follows for $|\bar{V}_k^n|$.

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